To Type A Mockingbird

Rick Statman
Carnegie Mellon University
Department of Mathematical Sciences
Pittsburgh, PA 15213,
e-mail: statman@cs.cmu.edu

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In his fascinating book "To Mock a Mockingbird", Ray Smullyan asks whether a fixed point combinator can be constructed from the Bluebird B, the Identity Bird I, and the Mockingbird M (sometimes called "little omega"). Here we show that this is not possible.

The B, I monoid was first singled out as a fragment of lambda calculus by Alonzo Church [2] in connection with the word problem. It comes up naturally as a subset of Church's basis B, I, C^*, W^*, K, where $M = W^* = WI$, and was further studied by Haskell Curry [3] in his work on the interdefinability of combinators; in particular, with reference to compositive, permutative, duplicative, and selective effect. This was taken up years later by the author [5].

A fixed point combinator F is a combinator satisfying

$$Fx = x(Fx).$$

Now B and M are sufficient to define fixed points

$$M(BxM) = BxM(BxM) = x(M(BxM))$$

and in the presence of W, we have a fixed point combinator

$$B(BMW)B.$$

But, without "permutative effect" it is unclear whether a fixed point combinator can be defined from B, I and M alone. Note that $BM(B(BM))B)$ has the right "Bohm tree" to be a fixed point combinator but it is not one. This can be proved directly.

The question was studied by Wos and McCune [9], and in [7] we proved that there is no such F satisying

$$Fx \twoheadrightarrow x(Fx)$$

where $\twoheadrightarrow$ is ordinary (weak beta) reduction. There, our proof used simple typed lambda calculus; here we use Girard's system of polymorphic lambda calculus to completely answer the question

We shall assume for the most part that the reader is familiar with classical lambda calculus and combinatory logic is in [2]. B, I, and M can be regarded as constants with (weak beta) reduction

$Ix \rightarrowtail x$
$Bxyz \rightarrowtail x(yz)$,
$Mx \rightarrowtail xx$

and the corresponding conversion relation, $\leftrightarrow$, or as lambda terms

$I = \lambda x.\, x$,
$B = \lambda xyz.\, x(yz)$, and
$M = \lambda x.\, xx$

with beta, or beta-eta reduction. Although Smullyan's question is about constants with weak-beta, it suffices to consider lambda terms with beta-eta, since these simulate the former, and the answer is negative. However, we will need to know a few things about $\leftrightarrow$.

Fact 1:

(i) $\rightarrowtail$ satisfies the standardization theorem.

(ii) $\leftrightarrow$ satisfies the Church-Rosser theorem.

Lemma 0.1. *Suppose that X is a combination of B, I and M and* $X\, x_1 \ldots x_k \leftrightarrow x_i X_1 \ldots X_l$.

Then $i = 1$ and X has $\rightarrowtail$ normal form.

Remark: This lemma is slightly stronger than the one stated in [7].

Proof. Suppose that $Xx_1 \dots x_k \leftrightarrow x_i X_1 \dots X_l$. By the Church-Rosser and standardization theorems

$$X\, x_1 \dots x_k \rightarrowtail\!\!\!\!\rightarrow x_i X_1 \dots X_l$$

by a head reduction (possibly for different X_j). The proof is by induction on the length of a head reduction. We first consider the case of a head reduction of X. This case follows immediately from the induction hypothesis. Next we consider the case that X is in head normal form

Case 1: $X = B, I, M$. This case is trivial.
Case 2: $X = BUV$. We have $BUVx_1 \dots x_k \rightarrowtail U(Vx_1)x_2 \dots x_k$. Now we can simulate the reduction

$$U(Vx_1)x_2 \dots x_k \rightarrowtail\!\!\!\!\rightarrow x_i X_1 \dots X_l$$

by $Ux_1x_2 \dots x_k$ until Vx_1 comes to the head or the reduction ends. But by the induction hypothesis the second alternative is impossible. Thus, by induction hypothesis U has a normal form. Now, when Vx_1 comes to the head, we have

$$Vx_1V_1 \dots V_m \rightarrowtail\!\!\!\!\rightarrow x_i X_1 \dots X_\ell$$

Now we can simulate this head reduction with

$$Vx_1y_1 \dots y_m$$

until one of the V_j comes to the head or the reduction ends. By the induction hypothesis the first alternative is impossible so by the induction hypothesis V has a normal form.

Case 3: $X = BU$. We have $BUx_1 \dots x_k \to U(x_1x_2)x_3 \dots x_k$. This case is similar to Case 2. □

It is worth noting that this lemma remains true if M is replaced by W. We need some notation

$$X^n x \;=:\; X(\dots(Xx)\dots) \qquad n \text{ occurrences of } X$$

$$X_n x \;=:\; ((\dots(XX)\dots)X)x \qquad n \text{ occurrences of } X,$$

and a definition.

Definition 0.1. *B^nB and B^nM are "monomials".*
Each monomial is a "polynomial".
If X and Y are polynomials then BXY is a polynomial.

Certain $\rightarrowtail$ normal terms like $B(Bxy)$ are not polynomials, but they are beta-eta convertible to polynomials.

Fact 2: $(B, I$ monoid)

$$\begin{array}{rlll}
(i) & Bx(Byz) & \text{beta} - \text{eta conv.} & B(Bxy)z \\
(ii) & BIx & \text{beta} - \text{eta conv.} & x \\
 & BxI & \text{beta} - \text{eta conv.} & x \\
(iii) & B(Bxy) & \text{beta} - \text{eta conv.} & B(Bx)(By)
\end{array}$$

Proof. Only (iii) may be unfamiliar.

$$B(Bx)(By)uv \to Bx(Byu)v \to x(Byuv) \to x(y(uv)) \leftarrow Bxy(uv) \leftarrow B(Bxy)uv.$$

□

Corollary 0.1. *If X is $\rightarrowtail$ normal then X beta-eta conv. to a polynomial.*

We shall use only a very restricted fragment of Girard's F_1. Types $a, b, c, \ldots$ are built up from type variables $p, q, r, \ldots$ and binary type relations symbols $P, Q, R, \ldots$ by $\to$ and $\forall$; viz, $p, q, r, \ldots$ are types if a, b are types then so are Rab, $a \to b$, and $\forall R\ a$. We also have terms which belong to the kind

$$\text{Type} \to (\text{Type} \to \text{Type}),$$

and these include $\lambda pq.p$ and $\lambda pq.q$. In short, we have no quantification over types. This system was suggested by Urzyczyn [8]. Here we shall make several conventions. First, we shall introduce and eliminate vacuous quantifiers at will, since this can be done at the term level by trivial type application and vacuous type abstraction. Second, we shall ignore the order of quantifiers in prefixes $R*$ for similar reasons. In particular, if we write $R*$ the sequence can be empty.

Definition 0.2. *A simple type $a_1 \to (\ldots(a_n \to q)\ldots)$ is said to be "quadratic" if*

(i) Each a_i has the form $p_1 \to (\ldots(p_m \to p)\ldots)$ where m depends on i (these are called "linear components ") and all the variables are distinct.

(ii) No variable occurs more than twice.

(iii) If a variable occurs twice then its first occurrence is positive and its second occurrence is strictly negative (we call this feature "decreasing").

(iv) If two variables in the same component both occur again later (as components) then they occur in the same order ("monotone").

Here q is said to be the "principal" variable.

Definition 0.3. *a is said to be "reflexive" if $a = (b \to c)$ and there exists $d*, e*$ ("reflectors") such that $[d * /p*]b = [e * /p*](b \to c)$. a is "hyper-reflexive" if $e*$ can be restricted to a change of variables.*

Definition 0.4. *A typing of X is said to be "tame" if it has the following properties*

(i) (Composition) Every B is typed

$$\forall P * (\forall R * S * (a \to b) \to \forall R * (\forall S * (c \to a) \to \forall S * (c \to b)))$$

(ii) (Semi-simple) The only non-trivial type applications are in the typings of occurrences of M.

(iii) (Distributive) If $X : a \to b$ and d is a strictly positive closure of b then there is a strictly positive closure c of a such that $X : c \to d$

Below we shall assert that several typings are tame. This will be easy to verify by inspecting the definitions.

Proposition 0.1. *Every polynomial X has a tame typing $X : a \to b$ where b is quadratic.*

Proof. First we type polynomials $X = X_1 @ \ldots @ X_k$ by recursion on k. For each k we have a subsidiary recursion on the length of X_1.

Basis: $k = 1$. Let $X = B(\ldots(BC)\ldots)$ with l explicit occurrences of B and $C = B$ or M. First assume that $l = 0$.

Case 1: $C = B$. Then we type $B : (p \to q) \to ((r \to p) \to (r \to q))$
Case 2: $C = M$. Then we type M:

$$\forall R(R(t \to p)t \to R(t \to q)p) \to (R(t \to p)t \to q) \to ((t \to p) \to q)$$

Recursion Step: Having typed $B(\ldots(BC)\ldots) : a \to b$ for $l - 1$ explicit B's, type $X : (a \to b) \to ((t \to a) \to (t \to b))$.

Primary Recursion Step: We suppose that we have typed

$$X_1@\ldots@X_k : a \to b$$

where b is a quadratic (simple) type, and $a = \forall R * c$ where c is quantifier free. We now wish to type $Y@X_1@\ldots@X_k$ where $Y = B(\ldots(BC)\ldots)$ with l explicit occurrences of B and $C = B$ or M. We do this by a subsidiary recursion l.

Basis:

Case 1: $C = B$. By hypothesis b is simple and w.l.o.g. we may assume that $b = c \to d$, for otherwise, we can substitute for the principle variable of b. We type $B : (c \to d) \to ((t \to c) \to (t \to d))$.
Case 2: $C = M$. By hypothesis b is quadratic, and thus hyperreflexive, with reflector $d*$, $e*$. Now we have the typings

$$X_1@\ldots@X_k : [e * /p*](a \to b)$$

$$X_1@\ldots@X_k : [d * /p*](a \to b)$$

and thus a typing

$$X_1@\ldots@X_k : [Rd * e * /p*](a \to b),$$

where we write $Rd * e*$ for $R\, d_1\, e_1 \ldots R\, d_n\, e_n$.
Hence, by the distributive property

$$X_1@\ldots@X_k : \forall R\, [Rd * e * /p*]a \to \forall R\, [Rd * e * /p*]b$$

but

$$Y : \forall R\, [Rd * e * /p*]b \to c$$

where b is a quadratic (simple) type. W.l.o.g. we can assume that $b = b_1 \to (\ldots(b_1 \to c)\ldots)$, possibly substituting a linear simple type for the principal variable of b. Now apply the construction at the basis case using the distributive property. □

Theorem 0.1. *There is no fixed point combinator.*

Proof. By contradiction. Suppose that we have B, I, M combination X such that Xx conv. $x(Xx)$. By Lemma 1, X has a $\rightarrowtail$ normal form. Thus, X beta-eta converts to a polynomial which we shall similarly denote X. Now, by the proposition, X has a type in F_1, hence, so does Xx. Thus, Xx has a beta-eta normal form, contradicting the fact that its Bohm tree is infinite. □

References

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