TLCA List of Open Problems

Updated February 4, 2014

Problem # 15

Submitted by <u>Alex Simpson</u>   *Date:* 1995

Statement. Is the equational theory of typed λ-calculus with finite sums the maximum consistent typically-ambiguous congruence?

Problem Origin.   The problem was posed by Alex Simpson and probably others.

Extend the simply-typed λ-calculus with syntax for finite product and sum types. This has a standard notion of β-equality, $=_\beta$. By $=_{\beta\eta}$ we mean the equational theory induced semantically by interpretations in bicartesian closed categories (cartesian closed categories with finite coproducts). It is not hard to axiomatize this equality. Following [Statman, 1983], a typed congruence relation $\sim$, is said to be *typically ambiguous* if it is preserved under type substitutions; that is: $s \sim t\colon \sigma$ implies $s^* \sim t^*\colon \sigma^*$, where the $(\cdot)^*$ transformation performs a substitution of types for atomic types. The main question is:

> *Is $=_{\beta\eta}$ maximum amongst consistent typically-ambiguous congruence relations containing $=_\beta$?*

A (possibly) weaker version of the question is simply: is it the case that the only typically-ambiguous congruence relation properly extending $=_{\beta\eta}$ is the inconsistent relation? There are various semantic equivalents to and consequences of a positive answer to this latter question. An equivalent statement is: given any bicartesian-closed category $\mathcal{C}$ that is not a preorder, the equational theory induced by all interpretations in $\mathcal{C}$ is exactly $=_{\beta\eta}$; see [Simpson, 1995] for a similar equivalence for the calculus without coproducts. In fact, it is enough to consider only the category of finite sets; that is, another equivalent is: the equational theory induced by interpretations in the category of finite sets is exactly $=_{\beta\eta}$. (To show that this implies that $=_{\beta\eta}$ is maximal amongst consistent typically-ambiguous congruences, one encodes the category of finite sets within the type theory using finite coproducts of the unit type.) The last equivalent above is simply the finite model property for $=_{\beta\eta}$, a standard consequence of which is decidability. A related result, the decidability of $=_{\beta\eta}$ in the case of non-empty coproducts is known from [Ghani, 1995] and [Altenkirch et al., 2001]. Again for non-empty sums only, the completeness of $=_{\beta\eta}$ with respect to interpretations in the category of sets was shown by [Dougherty and Subrahmanyam, 2000]; their argument makes essential use of infinite sets. A fragment of completeness relative to finite sets has been shown by [Altenkirch and Uustalu, 2004], who established completeness and decidability results for the extension of typed λ-calculus with a base type of booleans, but without other atomic types.

References

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