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Problem # 2 [SOLVED]

Submitted by Roger Hindley *Date:* Known since 1973!
Statement. Is ticket entailment decidable?
Problem Origin. The problem was first posed by Robert Meyer.

The question is whether there is a decision-algorithm for the implicational fragment $\mathrm{T}_{\rightarrow}$ of the propositional logic called *ticket entailment*. Equivalently, is there one for the simple type-theory of the restricted combinatory logic based on B, B', I, W? The logic $\mathrm{T}_{\rightarrow}$ has just one deduction-rule ($(\rightarrow\mathrm{E})$ or *modus ponens*), and four axiom-schemes:

$$(\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta)), \quad \alpha \rightarrow \alpha,$$
$$(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)), \quad (\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta).$$

Alternatively, let $\mathrm{CL}_{B,B',I,W}$ be the system of combinatory logic whose *terms* are built by application from four basic combinators with reduction rules:

$$\mathsf{B}XYZ \triangleright X(YZ), \quad \mathsf{B}'XYZ \triangleright Y(XZ), \quad \mathsf{I}X \triangleright X, \quad \mathsf{W}XY \triangleright XYY.$$

(Abstraction in $\mathrm{CL}_{B,B',I,W}$ is much weaker than in full combinatory logic; see [Trigg et al., 1994, §3] for a characterization by P. Trigg.) Let *types* be built by the operation $(\sigma \rightarrow \tau)$ from type-variables $a, b, c, ...$, and let types be assigned to terms as usual, starting from these four axiom-schemes:

$$\mathsf{B} : (\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta)), \quad \mathsf{I} : \alpha \rightarrow \alpha,$$
$$\mathsf{B}' : (\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)), \quad \mathsf{W} : (\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta).$$

Is there an algorithm that, when applied to any type τ, will decide whether there exists a term X in this system such that $X : \tau$ is provable?

System $\mathrm{T}_{\rightarrow}$ first appeared in print in [Anderson, 1960], although it dates back at least to work of Belnap in 1957. It was motivated and described in detail in [Anderson and Belnap, 1975, Chapter 1 §§ 6 and 8.3.2 (pp. 41–50 and 76)]. Its decidability question was first raised on p. 69 of that book. Proofs of the decidability and undecidability of several related systems were given in [Anderson et al., 1992, §§ 60–67 (pp. 267–391)]; for example in § 65.2 the logic T of ticket entailment was shown to be undecidable, but the method did not apply to its implicational fragment $\mathrm{T}_{\rightarrow}$. A decidability result for a restricted class of formulas can be found in [Broda et al., 2004].

Warnings: (1) In the 30 years since 1975 the $\mathrm{T}_{\rightarrow}$ problem and its combinatory equivalent have been tried by several very able workers without success. For example some relevant results are in [Bimbó, 2005] and [Bimbó, 2006].

(2) In papers on entailment, omitted parentheses are usually restored by "association to the left", not "to the right" as in types in type theory!

Solution: Two independent confluence proofs have been proposed in 2010. The solution by Katalin Bimbó and J. Michael Dunn is published in [Bimbó and Dunn, 2013]. The solution by Vincent Padovani is published in [Padovani, 2013].

References

[Anderson, 1960] Anderson, A. R. (1960). Entailment shorn of modality. *Journal of Symbolic Logic*, 25:388. (Abstract).

[Anderson and Belnap, 1975] Anderson, A. R. and Belnap, N. D. (1975). *Entailment. The Logic of Relevance and Necessity, Volume 1.* Princeton University Press, U.S.A.

[Anderson et al., 1992] Anderson, A. R., Belnap, N. D., and Dunn, J. M. (1992). *Entailment. The Logic of Relevance and Necessity, Volume 2.* Princeton University Press, U.S.A.

[Bimbó, 2005] Bimbó, K. (2005). Types of I-free hereditary right maximal terms. *Journal of Philosophical Logic*, 34:607–620.

[Bimbó, 2006] Bimbó, K. (2006). Relevance logics. In Jacquette, D., editor, *Philosophy of Logic*, volume 5 of *Handbook of the Philosophy of Science*, pages 723–789. North-Holland/ Elsevier, Amsterdam.

[Bimbó and Dunn, 2013] Bimbó, K. and Dunn, J. M. (2013). On the decidability of implicational ticket entailment. *Journal of Symbolic Logic*, 78(1):214–236.

[Broda et al., 2004] Broda, S., Damas, L., Finger, M., and e Silva, P. S. (2004). The decidability of a fragment of BB'IW-logic. *Theoretical Computer Science*, 318(3):373–408.

[Padovani, 2013] Padovani, V. (2013). Ticket entailment is decidable. *Mathematical Structures in Computer Science*, 23(3):568–607.

[Trigg et al., 1994] Trigg, P., Hindley, J. R., and Bunder, M. W. (1994). Combinatory abstraction using B, B' and friends. *Theoretical Computer Science*, 135:405–422.