TLCA List of Open Problems

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Problem # 3 [SOLVED]

Submitted by Richard Statman *Date:* 1981
Statement. Does the range property hold for the theory $\mathcal{H}$?
Problem Origin. The problem was first posed by Henk Barendregt.

Consider the least lambda theory $\mathcal{H}$ which equates all unsolvable terms. Take any combinator F. Prove that either $\mathcal{H} \vdash FM = FN$, for all closed M, N, or there are infinitely many closed M_i, such that no two of terms FM_i are equated under $\mathcal{H}$. This problem, known as the *range problem for* $\mathcal{H}$, is stated as Conjecture 20.2.8 in [Barendregt, 1984].

Comments by Richard Statman: Different researchers have different takes on this problem. Mine are the following.

(a) If we can compute functions recursive in the jump of 0 then the range property holds. The word problem for $\mathcal{H}$ is Σ_2-complete but there is no good notion of $\mathcal{H}$-computability that allows the "computation" of those number theoretic function recursive in jump of 0. If one knew such a notion of computability then one could copy the usual recursion theoretic proof of the range property for beta-eta.

(b) If we can construct solvable Plotkin terms then the range property fails. Take a Kleene enumerator E such that $E_0 = \Omega$ and E_{n+1} is the n-th solvable term. Then one can construct Plotkin terms F and G such that if $k > 0$ then

$$F_0(G_0)(E_0) \neq F_0(G_0)(E_k)$$
$$F_0(G_0)(E_k) = F_0(G_0)(E_{k+1})$$

This is for the beta-eta theory. If F and G can be constructed to be solvable then we would have a term with a range of size 2 for $\mathcal{H}$.

Solution: A negative solution has been presented by A. Polonsky in 2010 and published in [Polonsky, 2012], who constructs a term F with a two-element range. In contrast, Intrigila and Statman [Intrigila and Statman, 2011] showed that the range property for Combinatory Logic is true.

References

[Barendregt, 1984] Barendregt, H. (1984). *The Lambda Calculus. Its Syntax and Semantics.* North-Holland, second edition.

[Intrigila and Statman, 2011] Intrigila, B. and Statman, R. (2011). Solution to the range problem for combinatory logic. *Fundamenta Informaticae*, 111(2):203–222.

[Polonsky, 2012] Polonsky, A. (2012). The range property fails for H. *Journal of Logic and Computation*, 77(4):1195–1210.